

THE DISTANCE TRISECTOR CURVE IS TRANSCENDENTAL

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ABSTRACT. We show that the distance trisector curve is not an algebraic curve, as was conjectured in the founding paper by T. Asano, J. Matousek and T. Tokoyama [1].

1. INTRODUCTION

In the paper [1] (see also [2]) the curve called *the distance trisector curve* was introduced. It is defined as follows: Fix two points, say $P_1 = (0, 1)$ and $P_2 = (0, -1)$; then two curves, C_1 , C_2 , can be constructed such that they divide the plane in three sectors—hence the designation of the curves C_1 , C_2 as “trisectors,” in such a way that the distance between C_1 (the “upper” distance trisector curve) and C_2 (the “lower” curve) is the same as the distance from C_2 to P_2 ; the curve C_1 then determines a connected region around P_1 which can be thought of as the zone of influence of this point, and similarly for P_2 and C_2 , while the region between the two curves is a kind of “neutral zone,” and the curves C_1 , C_2 are symmetric with respect to the x axis.

Now, in the introduction of [1] the authors conjecture that “...*the distance trisector curve is not algebraic*...” —that is, that it cannot be expressed in the form $P(x, y) = 0$ for a polynomial $P \in \mathbb{R}[x, y]$, and this was probably the most important question left open in that work.

In this paper we show that this conjecture is true. The sketch of our proof is as follows:

First, based on results in [1], one knows that the distance trisector curve can be parametrized as $(t, f(t))$ where $f(t)$ is an analytic function. The important point for us here is that its Taylor expansion has coefficients in the quadratic field $\mathbb{Q}[\sqrt{3}]$, which in particular implies that if the distance trisector curve were an algebraic curve, expressed as $P(x, y) = 0$, then P would belong to $\mathbb{Q}[\sqrt{3}][x, y]$.

The field $\mathbb{Q}[\sqrt{3}]$ has a natural involution: the conjugation map $\sqrt{3} \rightarrow -\sqrt{3}$ in $\mathbb{Q}[\sqrt{3}]$. Technically, this is the only non-trivial automorphism in the Galois group of the extension $\mathbb{Q}[\sqrt{3}]$ of \mathbb{Q} , although this is not essential to our proof; the point is that from this another curve can be defined, which we call the *conjugate distance trisector curve*, or simply, the *conjugate curve*. This new curve shares several of the algebro-geometric properties of the distance trisector curve, but its geometric aspect is completely different as, roughly speaking, the shape of the conjugate curve resembles that of an Archimedean spiral.

But if the distance trisector curve were an algebraic curve then the conjugate curve would also be, because it could be expressed as $\overline{P}(x, y) = 0$, where $\overline{P} \in \mathbb{Q}[\sqrt{3}][x, y]$ is the conjugate polynomial. However, we can show that the conjugate curve, like the Archimedean spiral, has in fact an infinite number of crossings with the axes. Therefore, it cannot be an algebraic curve.

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2. SOME BASIC FACTS ABOUT THE DISTANCE TRISECTOR CURVE

This section is essentially an equivalent reformulation of some parts of [1].

2.1. The distance trisector curve as an envelope curve... First of all, we observe that the distance trisector can be seen as the envelope curve of a family of circles: Indeed, if we let $\alpha : I \rightarrow \mathbb{R}^2$ be a parameterization of the upper distance trisector curve, and for each $t \in I$, we let S_t be the circle centered at $\alpha(t)$ and of radius $d(\alpha(t), (0, 1))$, then the lower distance trisector curve is the envelope of the family of circles S_t .

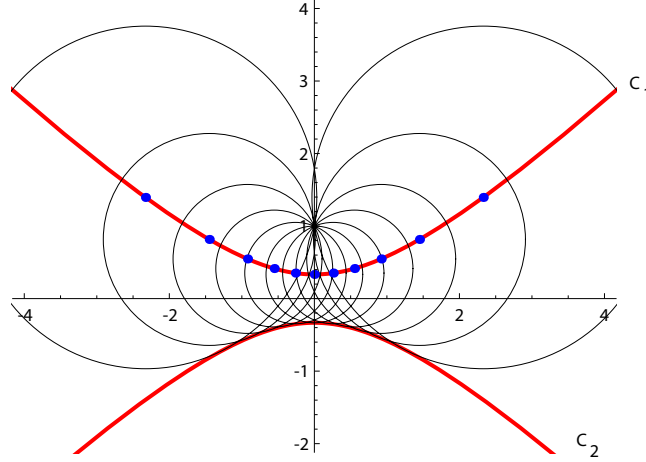


Figure I. The two distance trisector curves and some of the circles S_t .

For later use, let us next recall how to obtain in general the parametrization of the envelope curve of a family of circles defined as before.

Proposition 1. *Let $\alpha : I \rightarrow \mathbb{R}^2$, $I \subset \mathbb{R}$, be a regular parametrized curve not passing through the point $(0, 1)$. For each $t \in I$, let S_t be the circle centered at $\alpha(t)$ and of radius $d(\alpha(t), (0, 1))$. The envelope curve of the family of circles $\{S_t\}_{t \in I}$ can be parametrized by*

$$\beta(t) = (0, 1) + 2\langle \alpha(t) - (0, 1), \vec{n}(t) \rangle \vec{n}(t),$$

where \vec{n} denotes the normal vector to the curve α .

Moreover, the tangent line to the envelope β at t_0 only depends on $\alpha(t_0)$ and on the tangent line to α at t_0 .

Proof. Let us suppose that $\alpha(t) = (a(t), b(t))$, then the implicit equation of the circle S_t is

$$(2.1) \quad F(x, y, t) := -1 + x^2 + y^2 - 2xa(t) - 2(-1 + y)b(t) = 0.$$

Thus, if we solve the system of equations

$$\begin{cases} F(x, y, t) &= -1 + x^2 + y^2 - 2xa(t) - 2(-1 + y)b(t) = 0, \\ F_t(x, y, t) &= -2xa'(t) - 2(-1 + y)b'(t) = 0, \end{cases}$$

in the variables x, y we obtain a parameterization of the envelope of the family of circles.

Indeed, from the second equation we have that

$$\frac{x}{1 - y} = \frac{b'(t)}{a'(t)}.$$

Substituting now $x = (1 - y) \frac{b'(t)}{a'(t)}$ into the first equation we get a quadratic equation for y

$$\frac{1 - y}{(a'(t))^2} \left(-((a'(t))^2 + (b'(t))^2) y - (a'(t))^2 + (b'(t))^2 - 2a'(t)b'(t)a(t) + 2b(t)(a'(t))^2 \right) = 0,$$

and it follows that

$$\begin{aligned}
\beta(t) &= \frac{1}{a'(t)^2 + b'(t)^2} (2b'(t)((1 - b(t))a'(t) + a(t)b'(t)), (2b(t) - 1)a'(t)^2 - 2a(t)a'(t)b'(t) + b'(t)^2) \\
&= \frac{1}{\|\alpha'(t)\|^2} (2((b(t) - 1)a'(t) - a(t)b'(t))(-b(t), a(t)) + (0, \|\alpha'(t)\|^2)) \\
&= 2\langle (a(t), b(t) - 1), \frac{(-b(t), a(t))}{\|\alpha'(t)\|} \rangle \frac{(-b(t), a(t))}{\|\alpha'(t)\|} + (0, 1) \\
&= (0, 1) + 2\langle \alpha(t) - (0, 1), \vec{n}(t) \rangle \vec{n}(t),
\end{aligned}$$

as stated.

For the second assertion, we simply compute the tangent vector to the curve β , and the last expression shows that this is independent of the parameterization of the initial curve α ; to simplify matters, it is better to work with the arc-length parameterization; then, a simple computation shows that

$$\beta'(s) = -2\kappa(s) \left(\langle \alpha(s) - (0, 1), \vec{t}(s) \rangle \vec{n}(s) + \langle \alpha(s) - (0, 1), \vec{n}(s) \rangle \vec{t}(s) \right),$$

where \vec{t} denotes the tangent vector to the curve α and κ its curvature function. \square

Remark 1. *Later on we will need the unit tangent vector to the curve β . Therefore, let us compute first $\|\beta'\|$:*

$$\begin{aligned}
\|\beta'(s)\|^2 &= 4\kappa^2(s) \left(\langle \alpha(s) - (0, 1), \vec{t}(s) \rangle^2 + \langle \alpha(s) - (0, 1), \vec{n}(s) \rangle^2 \right) \\
&= 4\kappa^2(s) \|\alpha(s) - (0, 1)\|^2.
\end{aligned}$$

Thus,

$$\vec{t}^\beta(s) = -\left\langle \frac{\alpha(s) - (0, 1)}{\|\alpha(s) - (0, 1)\|}, \vec{t}(s) \right\rangle \vec{n}(s) - \left\langle \frac{\alpha(s) - (0, 1)}{\|\alpha(s) - (0, 1)\|}, \vec{n}(s) \right\rangle \vec{t}(s).$$

2.2. Comparison with Lemma 8 in Assano *et al.* The important point about stating the previous proposition is that it leads to consider a map Θ that, given a parametrized curve $\alpha(t) = (a(t), b(t))$, transforms it to another parametrized curve

$$\Theta(\alpha)(t) = 2\langle \alpha(t) - (0, 1), J \left(\frac{\alpha'(t)}{\|\alpha'(t)\|} \right) \rangle (T \circ J) \left(\frac{\alpha'(t)}{\|\alpha'(t)\|} \right) - (0, 1),$$

where T is the symmetry $T(x, y) = (x, -y)$ and J is the rotation $J(x, y) = (-y, x)$. It is actually plain that, by definition, the distance trisector curve is characterized by the following property: it is a curve such that if $\alpha(t)$ is a local parametrization then $\Theta(\alpha)(t)$ is another local parametrization of the same curve; that is

$$\Theta(\alpha)(t) = \alpha(g(t)), \quad \forall t \in \mathbb{R},$$

with $g : \mathbb{R} \rightarrow \mathbb{R}$ the reparametrization.

To facilitate comparisons with [1], we now observe that in that work the trisector curve is described as the graph of a function f defined on the whole real line \mathbb{R} , $\alpha(x) = (x, f(x))$. In other words, the parameter t is chosen to be x , and the coordinate functions $a(t) \rightarrow x$ and $b(t) \rightarrow f(x)$; according to the previous paragraph, this can be written as

$$\Theta(\alpha)(x) = (t(x), f(t(x))),$$

where the reparametrization g has been denoted by t .

Let us next recall the following basic result from [1]:

Lemma 1. (Lemma 8 of [1]) *The following equations are satisfied for every $x \in \mathbb{R}$:*

$$(2.2) \quad \begin{aligned} (t(x) - x)^2 + (f(t(x)) + f(x))^2 - x^2 - (f(x) - 1)^2 &= 0, \quad \text{and} \\ t(x) - x + (f(x) + f(t(x)))f'(t(x)) &= 0, \end{aligned}$$

where $f'(t(x))$ is the derivative of f evaluated at $t(x)$.

Remark 2. *The first equation in (2.2) is just $F(t(x), -f(t(x)), x) = 0$ (see Eq. (2.1) for the definition of F). The second equation in (2.2) comes from the fact that for a fixed x , the point $(t(x), -f(t(x)))$ minimizes the squared distance of $(x, f(x))$ to $(u, -f(u))$ among all u . Therefore,*

$$0 = \frac{1}{2} \frac{\partial}{\partial u} \Big|_{u=t(x)} ((u - x)^2 + (f(x) + f(u))^2) = t(x) - x + (f(x) + f(t(x)))f'(t(x)).$$

By the way, it is perhaps worthwhile of notice that in [1] this equation appears without two parentheses. We believe this to be a misprint, later corrected in Eq. (4) of the paper.

3. THE CONJUGATE DISTANCE TRISECTOR CURVE

In [1] the authors are looking for a **convex** curve, because intuitively this is the shape of the distance trisector curve in a neighborhood of its initial point $(0, 1/3)$. Nevertheless, the convexity hypothesis is not essential for the algebraic manipulations, and can be suppressed. Somewhat surprisingly, another curve appears sharing with the distance trisector curve many of its properties; we have called this new curve the conjugate distance trisector curve, or for brevity, the *conjugate* curve. Let us now elaborate on this point:

3.1. Power series for f and t near the origin. Along the proof on Lemma 10 in [1], which is the technical result needed to compute the Taylor series expansion of the distance trisector curve, the authors arrive to a point where a solution for the equations

$$(3.1) \quad \lambda - 1 + \frac{4}{3}\lambda q_2 = 0, \quad \lambda^2 - 2\lambda + \frac{4}{3}(\lambda^2 + 2)q_2 = 0,$$

has to be found; here λ and q_2 are the unknowns. Now, equations (3.1) are easily seen to have the two different sets of solutions:

$$(3.2) \quad \begin{cases} \lambda = +\sqrt{3} - 1, & q_2 = \frac{3}{8}(+\sqrt{3} - 1), \\ \lambda = -\sqrt{3} - 1, & q_2 = \frac{3}{8}(-\sqrt{3} - 1), \end{cases}$$

and notice that the only difference between them is the sign of $\sqrt{3}$.

In any case, choosing one of these solutions, the rest of the coefficients in the Taylor series expansion can be obtained recursively, as solutions of linear systems where all the coefficients belongs to the field $\mathbb{Q}[\sqrt{3}]$. In particular, all the coefficients in the Taylor expansion belong to this field. The choice made in [1] is then the first set of solutions, which gives raise to a convex curve.

But the second set of solutions is of course also possible, and this gives raise to a concave solution, which in a sense is a conjugation of the previous curve. In other words, its Taylor expansion is essentially the same as the Taylor expansion of the distance trisector curve, but with $\sqrt{3}$ replaced by $-\sqrt{3}$. Therefore the proof of the convergence of the new series is completely analogous to the existing one. (For instance, in this case the determinant d_k of the matrix of coefficients of the linear system is always negative, and the maximum value is $d_4 = -497.415$, so the minimum absolute value is 497.415.) The first few terms in the Taylor expansions for the new solution are in fact

$$f(x) = \frac{1}{3} - \frac{3}{8} \left(1 + \sqrt{3}\right) x^2 - \frac{27}{704} \left(13 + 7\sqrt{3}\right) x^4 + O(x^6),$$

$$t(x) = -(1 + \sqrt{3})x + \frac{27}{88} (17 + 10\sqrt{3}) x^3 + O(x^5).$$

3.2. Extending to all of \mathbb{R} . The main difference between the distance trisector curve and its conjugate curve is that the latter is not the graph of a function, as this curve has self-intersections.

To justify this, we notice that although in principle the functions f and t for the conjugate curve are known to be analytic only on some neighborhood of 0, in fact Lemma 11 in [1] can again be used to extend this neighborhood iteratively.

These assertions can be nicely illustrated using an approximation to the conjugate curve —obtained from Proposition 1— as follows: take $\alpha_0(t) = (t, \frac{1}{3} - t^2)$, $t \in [-1, 1]$, and define recursively $\alpha_{i+1} = \Theta(\alpha_i)$ for $i > 0$. It turns out that $\alpha_5(t)$ already gives a very good approximation in an extended interval to the conjugate curve, and as shown in Fig. II, it has a shape resembling an Archimedean spiral.

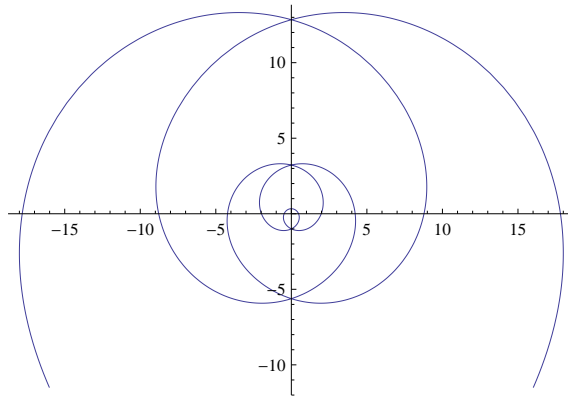


Figure II. The conjugate distance trisector curve.

4. SOME PROPERTIES OF THE CONJUGATE DISTANCE TRISECTOR CURVE.

4.1. The reflected curve. As said in Section 2, the lower distance trisector curve can be seen as the envelope curve of a family of circles centered at points of the upper curve. Due to the fact that it has been defined through Eqs. (2.2), the same happens to the conjugate distance trisector curve (see Fig. III).

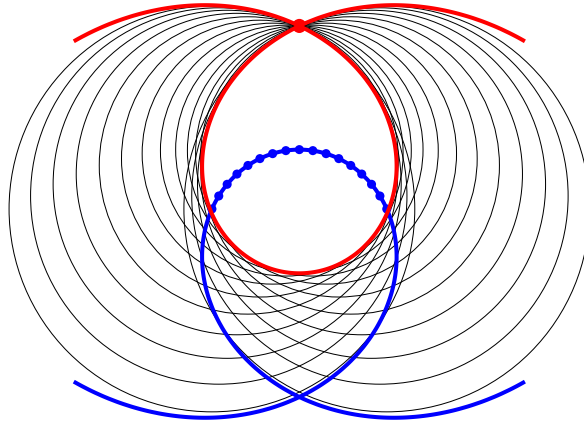


Figure III. The conjugate distance trisector curve (blue), its reflected curve (red) and some of the circles S_t .

In this case it makes little sense to speak of “upper” and “lower” curves, and thus we will rather refer to them as the conjugate curve and its reflection (with respect to the x -axis).

This result also shows that for any point $\alpha(t)$ of the conjugate distance trisector curve there is indeed a point $\beta(t) = T(\Theta(\alpha)(t))$ of the reflected curve such that the distance between $\alpha(t)$ and $\beta(t)$ is the same as the distance between $\alpha(t)$ and $p = (0, 1)$.

Moreover, this brings up another important difference between the two curves; namely, that if $\alpha(t_0)$ is a point of the conjugate curve and $\beta(t_0) = \alpha(g(t_0))$ is its corresponding point on the reflected curve, then we can only assure that this point *locally* minimizes the squared distance from $\alpha(t_0)$ to $\beta(t)$.

As an explicit example, take $t_0 = \frac{1}{32}$, so that $\alpha_5(t_0) = (0.92795, 2.82373)$ and $\Theta(\alpha_5)(t_0) = (2.2336, -4.39928) = \alpha_5(-0.0858323)$.

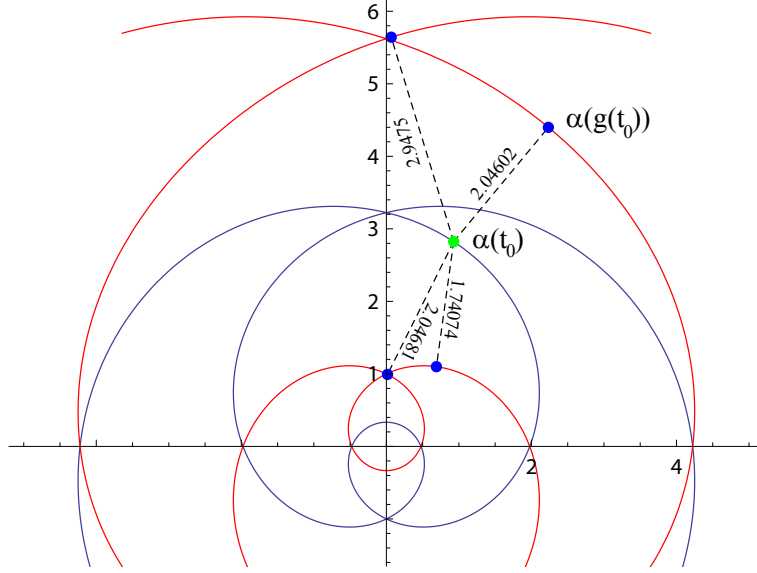


Figure IV. Visualization of the first four local minima of the distance function $\|\Theta(\alpha_5)(\frac{1}{32}) - \alpha_5(t)\|$.

The function $\|(2.2336, 4.39928) - \alpha_5(t)\|^2$ then has a local minimum (4.18942) at $t = -0.0858323$, but it has a global minimum (3.03018) at $t = 0.0134386$ (see Fig. IV).

Remark 3. As can be seen from Fig. IV, it is only the point marked $\alpha(g(t_0))$ that satisfies the property that its distance to the conjugate curve is the same as the distance from this curve to the point $(0, 1)$, and thus this is the point that belongs to the circle S_{t_0} of the envelope construction.

The slight discrepancies between the distances (less than $1/1000$) reflect the fact that α_5 is only an approximation to the conjugate curve, but also show that it is indeed a very good one.

Another property of the distance trisector curve, of perhaps still more interest here, is the following easy consequence of Proposition 1 (see also [1]).

Corollary 1. If α is a parametrization of the distance trisector curve, then the segment joining the point $(0, 1)$ with $T(\Theta(\alpha)(t))$ is parallel to the normal vector to α at $\alpha(t)$.

A graphical interpretation of the statement of Corollary 1 is given in Fig. V, and it is then clear that the same property is still valid for the conjugate curve (see Fig. VI).

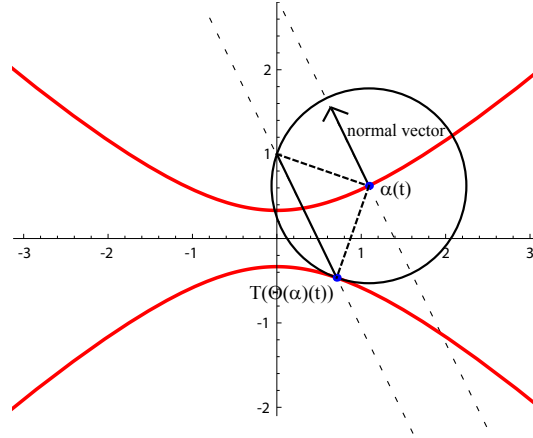


Figure V. A graphical interpretation of the statement of Corollary 1.

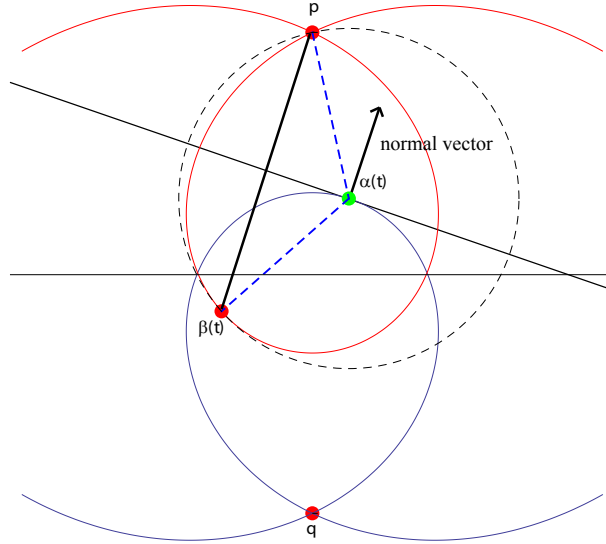


Figure VI. The conjugate distance trisector curve (blue) and its horizontal reflection (red) also satisfy Corollary 1.

An important point to stress here is that the tangent line at $\alpha(t)$ is the bisectrix of the angle $\widehat{P\alpha(t)\beta(t)}$.

4.2. The conjugate curve has self-intersections. Let us now prove the existence of the crossings in the conjugate curve:

Proposition 2. *The conjugate distance trisector curve passes through $(0, -1)$. Therefore it has a self-intersection at this point.*

Proof. It is clear that at the point $(0, \frac{1}{3})$ the tangent line to the conjugate distance trisector curve is horizontal. However, and in contrast to the trisector curve, it is not difficult to see, even for the local Taylor series expansion, that there is a point where the tangent line is vertical. Indeed, numerical computations show that the first point where the tangent line is vertical is approximately $(0.524251, -0.243883)$.

Now, as the tangent line runs, either to the right or to the left, from the horizontal position to the vertical position, there is a point where it passes through the point $(0, 1)$.

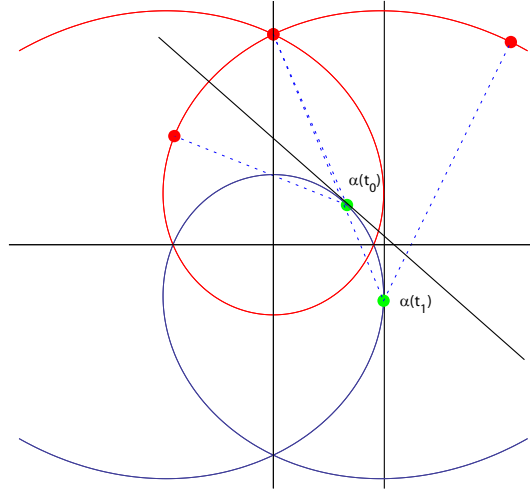


Figure VII. When the tangent line at $\alpha(t_0)$ moves to the tangent line at $\alpha(t_1)$, there is a point where it passes through the point $(0, 1)$.

Such a point exists because if we assign, say, a positive value to the angle $\widehat{P\alpha(t)\beta(t)}$ at a point such as $\alpha(t_0)$, then the angle at $\alpha(t_1)$ is negative and therefore at some point it has the value 0 (see Fig. VII).

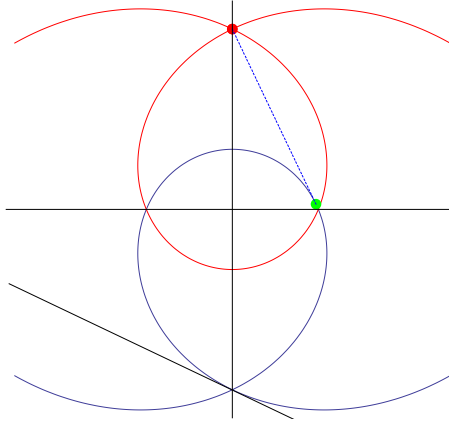


Figure VIII. The tangent line at $(0.464045, 0.0289289)$ passes through $(0, 1)$.

But if the reflected curve passes through $(0, 1)$, the conjugate curve passes through $(0, -1)$. Since the curve is symmetric with respect to y axis, this point is a self-intersection of the curve. \square

Remark 4. *Later on we will need a tangent vector at $(0, -1)$. A numerical approximation shows that one of the tangents at this point is generated by the unit vector $(0.902272, -0.431168)$ (see Fig. VIII).*

4.3. Horizontal tangent lines of the conjugate curve. More generally, horizontal tangent lines are associated to crossings of the conjugate curve with the y -axis:

Lemma 2. *Suppose $\alpha(t_0)$ is a point on the conjugate distance trisector curve with horizontal tangent line; then $T(\beta(t_0))$ is a point on the y -axis, whose tangent line is parallel to $J(\alpha(t_0) - (0, 1))$.*

Proof. Recall that J is rotation through an angle of $\pi/2$, and T reflection with respect to the x axis. Let us also recall that the tangent vector to the curve

$$\beta(t) = (0, 1) + 2\langle \alpha(t) - (0, 1), \vec{n}(t) \rangle \vec{n}(t),$$

is given by

$$\vec{t}^\beta(t) = -\left\langle \frac{\alpha(t) - (0, 1)}{\|\alpha(t) - (0, 1)\|}, \vec{t}(t) \right\rangle \vec{n}(t) - \left\langle \frac{\alpha(t) - (0, 1)}{\|\alpha(t) - (0, 1)\|}, \vec{n}(t) \right\rangle \vec{t}(t).$$

Now, if $\alpha(t_0)$ is a point on the conjugate distance trisector curve with horizontal tangent line, then

$$\vec{t}(t_0) = J(\vec{n}(t_0)) = (1, 0), \quad \vec{n}(t_0) = (0, 1).$$

Thus,

$$\beta(t_0) = (0, 1) + 2\langle (x(t_0), y(t_0) - 1), (0, 1) \rangle (0, 1) = (0, 2y(t_0) - 1),$$

and therefore, $T(\beta(t_0)) = (0, -2y(t_0) + 1)$ is a point where the conjugate curve crosses the y axis. We will refer to this point as the *crossing point associated to* $\alpha(t_0)$

Moreover, if we write

$$\frac{\alpha(t_0) - (0, 1)}{\|\alpha(t_0) - (0, 1)\|} = (a, b),$$

then

$$\vec{t}^\beta(t_0) = -(b, a),$$

and

$$T(\vec{t}^\beta(t_0)) = (-b, a) = J(a, b).$$

□

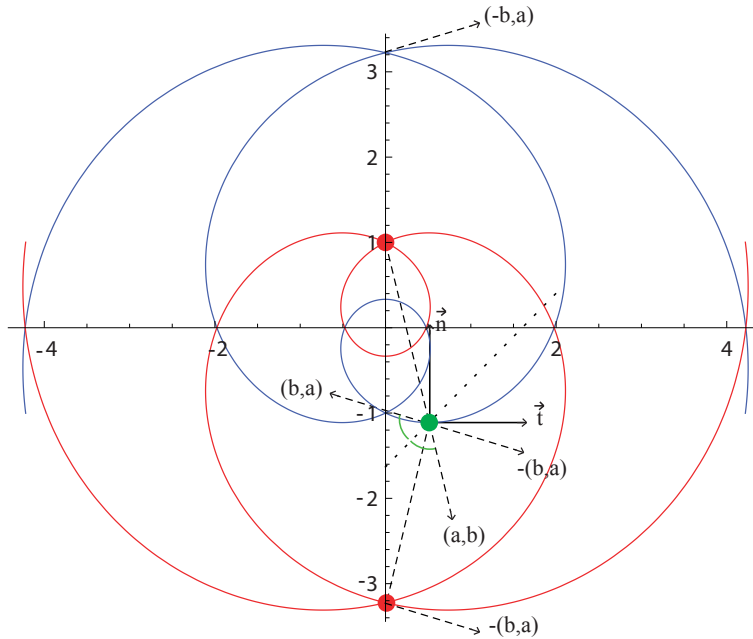


Figure IX. Visualization of the statement of Lemma 2.

A consequence of this is that when $\alpha(t_0)$ is below the line $y = -1$, then the associated crossing point is above the line $y = 1$ and one of the tangent vectors to the conjugate curve at that point has both coordinates strictly positive, as shown in Fig. IX. Clearly, an analogous result holds true when $\alpha(t_0)$ is over the line $y = 1$. In this case, the associated crossing point is under the x -axis and a tangent vector can be taken with both coordinates strictly negative.

5. IF THE DISTANCE TRISECTOR CURVE WERE AN ALGEBRAIC CURVE...

We have seen in Section 3.1 that the distance trisector curve can be parametrized as $\alpha(t) = (t, f(t))$, where the function f is analytic and its Taylor expansion has coefficients in $\mathbb{Q}[\sqrt{3}]$. To make the connection with the aim of this paper we now prove:

Lemma 3. *If the distance trisector curve were an algebraic curve, defined by an implicit equation $P(x, y) = 0$ with $P \in \mathbb{R}[x, y]$, then we can assume that $P \in \mathbb{Q}[\sqrt{3}][x, y]$.*

Proof. We can suppose that P is an irreducible polynomial of total degree $\leq n$. Also, for any non zero real number a , the implicit equation $(aP)(x, y) = 0$ defines the same algebraic curve; but, by irreducibility of P , if we fix any non vanishing coefficient in the polynomial, then there is just one possible implicit equation.

Also, since $\alpha(0) = (0, \frac{1}{3}) = (0, f(0))$, it is better to write the polynomial P in terms of the power basis $\{x^i(y - \frac{1}{3})^j\}_{i,j}$:

$$P(x, y) = \sum_{i+j \leq n} \frac{p_{i,j}}{i!j!} x^i \left(y - \frac{1}{3}\right)^j, \quad p_{i,j} \in \mathbb{R}.$$

We will now show that all the coefficients $p_{i,j}$ can be chosen in $\mathbb{Q}[\sqrt{3}]$.

Let us write the Taylor expansion of f as

$$f(t) = \sum_{i \in \mathbb{N}} m_i t^i = \frac{1}{3} + m_1 t + m_2 t^2 + \dots,$$

where $m_i \in \mathbb{Q}[\sqrt{3}]$. Furthermore, since f is obviously an even function, we can suppose that $m_{2k+1} = 0$ for $k \in \mathbb{N}$.

From the equation $P(x, y) = 0$ we have that $P(t, f(t)) = 0$. Therefore, the derivatives also satisfy $\frac{d^k}{dt^k} |_{t=0} P(t, f(t)) = 0$, for any $k \in \mathbb{N}$.

The first derivative is

$$\frac{d}{dt} |_{t=0} P(x(t), y(t)) = P_x(0, \frac{1}{3}) + P_y(0, \frac{1}{3}) f'(0) = P_x(0, \frac{1}{3}) = p_{1,0},$$

since $f'(0) = m_1 = 0$. Therefore, $p_{1,0} = 0$.

The second derivative is

$$\begin{aligned} \frac{d^2}{dt^2} |_{t=0} P(t, f(t)) &= P_{xx}(0, \frac{1}{3}) + 2P_{xy}(0, \frac{1}{3}) f'(0) + P_{yy}(0, \frac{1}{3}) (f'(0))^2 + P_y(0, \frac{1}{3}) f''(0) \\ &= p_{2,0} + p_{0,1} m_2, \end{aligned}$$

and so on.

Thus, in general the condition $\frac{d^k}{dt^k} |_{t=0} P(x(t), y(t)) = 0$ can be written as a homogeneous linear equation in the unknowns $p_{i,j}$, where the coefficients are computed from m_i through sums or products; and since we have supposed that the curve is algebraic, there are solutions to all these equations. We can moreover suppose that there is a coefficient of P

equal to 1, for if $p_{i_0, j_0} \neq 0$, then the i_0, j_0 coefficient of $\frac{P}{p_{i_0, j_0}}$ is 1. But assuming this, the solution is unique, because the polynomial P is irreducible.

Since all the coefficients in the system belong to $\mathbb{Q}[\sqrt{3}]$, the same holds for its solution $\{p_{i,j}\}_{0 \leq i+j \leq n}$. Thus, $P \in \mathbb{Q}[\sqrt{3}][x, y]$, as stated. \square

Now, the conjugation map: $a + b\sqrt{3} \rightarrow a - b\sqrt{3}$ in the field $\mathbb{Q}[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$ extends to the polynomial ring $\mathbb{Q}[\sqrt{3}][x, y]$. And if an algebraic curve is defined by an equation $P(x, y) = 0$ with $P \in \mathbb{Q}[\sqrt{3}][x, y]$, then the conjugate polynomial, \bar{P} , also defines an algebraic curve. Therefore and applying this to our case, if the distance trisector curve were an algebraic curve, its conjugate curve would be algebraic too.

6. THE CONJUGATE DISTANCE TRISECTOR CURVE IS NOT AN ALGEBRAIC CURVE

The main technical result is now:

Lemma 4. *There is an infinite number of intersections between the y -axis and the conjugate distance trisector curve.*

Proof. Let $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$, $\alpha(t) = (x(t), y(t))$, be a regular parametrization of the conjugate curve. We are going to construct a sequence $\{C_n = \alpha(t_n) = (0, y(t_n))\}_{n \in \mathbb{N}}$ of crossing points such that

$$\begin{aligned} t_0 &= 0, & C_0 &= \alpha(0) = (0, \frac{1}{3}), \\ t_1 &> 0, & C_1 &= \alpha(1) = (0, -1), \\ t_n &< t_{n+1}, & 0 &\notin x([t_n, t_{n+1}]), \\ \text{sg}(y(t_n)) &= (-1)^n, & |y(t_n)| &< |y(t_{n+1})|, & \text{and } |y(t_n) - 1| &\geq 2^{n-1} \quad (n > 0). \end{aligned}$$

We will use some auxiliary sequences: A sequence

$$\{V_n = \alpha(s_n) = (x(s_n), y(s_n))\}_{n \in \mathbb{N}}$$

of points with vertical tangent line—that is, $x'(s_n) = 0$, such that

$$x(s_n) < x(s_{n+1}), \quad x(s_n) > 2^n,$$

and a sequence

$$\{H_n = \alpha(u_n) = (x(u_n), y(u_n))\}_{n \in \mathbb{N}}$$

of points with horizontal tangent line—that is $y'(u_n) = 0$, such that

$$|y(u_n)| < |y(u_{n+1})|, \quad |y(u_n)| > 2^n,$$

and such that

$$t_n < u_n < s_n < t_{n+1}.$$

Finally, we will need a sequence $\{P_n = \alpha(r_n)\}$ of points where the conjugate curve crosses the line $y = 1$.

Throughout the proof, points on the reflected curve will be marked with a \sim , whereas the corresponding points on the conjugate curve will go without the \sim .

Obviously, $H_0 = (0, 1/3)$, and the existence of the points $V_0 = (x(s_0), y(s_0))$ and $P_0 = C_1 = (0, -1)$ was established in Proposition 4.2

Now, since V_0 is between C_0 and C_1 , then

$$t_0 = 0 < s_0 < t_1.$$

And from what has been said, such a point is related to another point in the reflected curve, \bar{P}_1 , with second coordinate $= 1$ (see Fig. X, left).

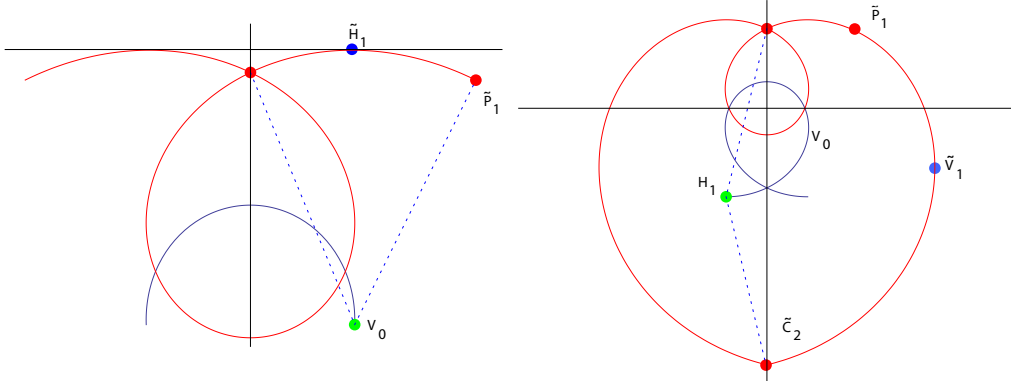


Figure X. Construction of the first points where the tangent lines are horizontal or vertical .

Therefore, between $(0, 1)$ and \tilde{P}_1 there should be a point, $\tilde{H}_1 = (x(u_0), y(u_0))$, in the reflected curve with horizontal tangent, and similarly for the conjugate curve. Let us call the latter H_1 . Notice that $t_1 < u_1$ and that the absolute value of the second coordinate of \tilde{H}_1 is greater than 1, because at $(0, 1)$ the reflected curve has a tangent vector with second coordinate positive (see Remark 4).

Since H_1 has an horizontal tangent line, it has associated a new crossing point in the reflected curve $\tilde{C}_2 = (0, -y(t_2))$, and Proposition 4.2 also gives that $-y(t_2) < -3$. Therefore $|y(t_2) - 1| = y(t_2) - 1 > 3 - 1 = 2$.

Now, between \tilde{P}_1 and \tilde{C}_2 , there is a point on the reflected curve with vertical tangent, $\tilde{V}_1 = (x(s_1), y(s_1))$, with $u_1 < s_1 < t_2$. And by a similar reasoning as the one in Proposition 4.2, its first coordinate is greater than twice the first coordinate of V_0 . (see Fig. X, right).

We can then iterate the process: \tilde{V}_1 generated the crossing \tilde{C}_2 , but because of the envelope construction of the reflected curve, V_1 also generates a point \tilde{P}_2 ; and between \tilde{V}_1 and \tilde{P}_2 , there is also a point in the reflected curve with horizontal tangent, \tilde{H}_2 , so we have the corresponding points in the conjugate curve, and so on, as shown in Fig. XI:

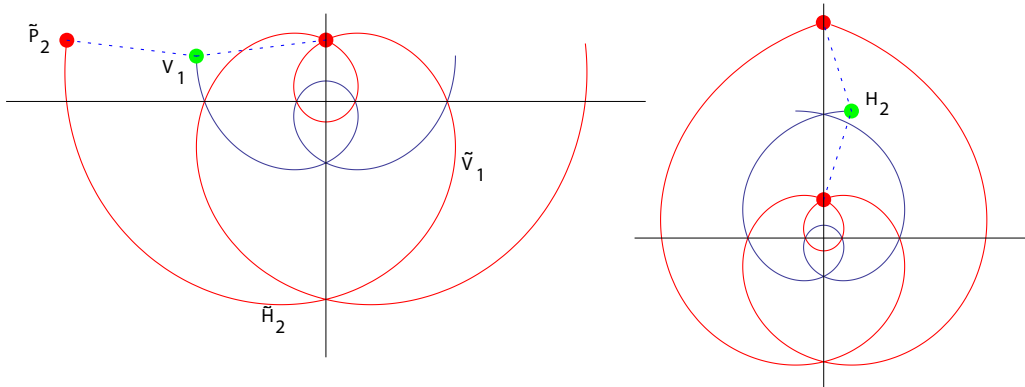


Figure XI. Construction of the next points where the tangent lines are horizontal or vertical .

Let us now state our induction hypothesis; we will suppose that $y(t_n) > 0$, the other case being analogous:

Between $C_{n-1} = (0, y(t_{n-1}))$ and $C_n = (0, y(t_n))$ there is a point $V_{n-1} = (x(s_{n-1}), y(s_{n-1}))$ with $t_{n-1} < s_{n-1} < t_n$. This point generates another point \tilde{P}_n in the reflected curve, whose second coordinate is less than the second coordinate of C_n .

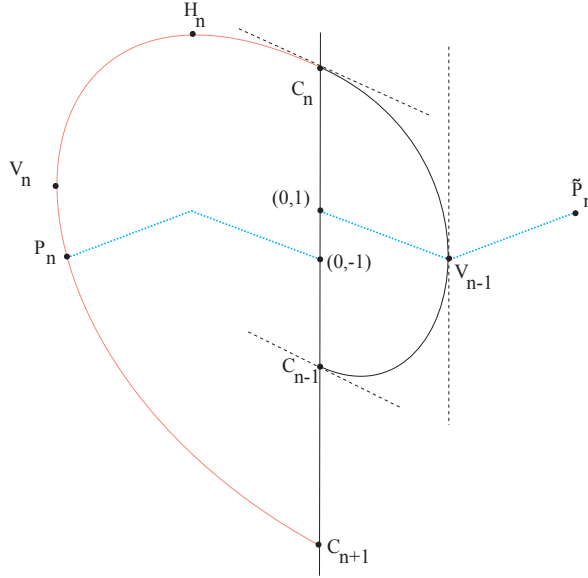


Figure XII. Inductive proof of the existence of the crossing point C_{n+1} .

The point \tilde{P}_n then defines a point P_n in the conjugate curve. Since the conjugate curve goes from C_n to P_n , and since the tangent line at C_n goes up, then there is a new point $H_n = (x(u_n), y(u_n))$ with $y(t_n) < y(u_n)$ and $t_n < u_n$. This new point defines another new crossing point $\tilde{C}_{n+1} = (0, -y(t_{n+1}))$ in the reflected curve, and thereby a point $C_{n+1} = (0, y(t_{n+1}))$ in the conjugate curve, and so on.

The fact now is that, just as in Proposition 4.2, $-y(t_{n+1}) = 1 + 2(y(u_n) - 1) > 1 + 2(y(t_n) - 1) = 2y(t_n) - 1$. Thus,

$$|y(t_{n+1}) - 1| = -y(t_{n+1}) + 1 > 2y(t_n) - 1 + 1 > 2 \times 2^{n-1} = 2^n.$$

Finally, since the conjugate curve goes from H_n to C_{n+1} , then there is a new point $V_n = (x(s_n), y(s_n))$, with $u_n < s_n < t_{n+1}$, whose distance to the y -axis is greater than the distance from P_n to the y -axis. This distance is greater than twice the distance from V_{n-1} to the y -axis.

In particular, all the crossing points are distinct. \square

We are now ready to conclude:

Theorem 1. *The distance trisector curve is a transcendental curve.*

Proof. As already mentioned, if the distance trisector curve were an algebraic curve, then its conjugate curve would be an algebraic curve too.

But since the number of intersections of the conjugate curve with the y -axis is infinite, this curve cannot be algebraic because, according to Bézout's Theorem, the number of intersections between any two algebraic curves is always finite.

Therefore, the distance trisector curve is transcendental, as claimed. \square

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